Eur. Phys. J. A **7**, 225–228 (2000)

THE EUROPEAN PHYSICAL JOURNAL A

Multifractal critical thermodynamics in nucleus-nucleus collisions

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Received: 19 October 1999
Communicated by B. Povh

Abstract. Critical points approach in the frames of multifractal thermodynamics is suggested to interpret the experimental data on nuclear multifragmentation which come from interactions in nuclear emulsion (in which $^{197}_{79}Au_{118}$ nuclei of energy 1 GeV/nucleon break up into fragments) and from the charge distributions of projectile fragments in sulphur (^{32}S) fragmentation at 200 GeV/nucleon. It is also shown that multifragmentation after macro-solids collisions exhibits properties analogous to those observed in the nuclear multifragmentation experiments.

PACS. 25.70.Pq Multifragment emission and correlations

Intermittency in multiparticle production at hadron-hadron and hadron-nucleus interactions is now well known phenomenon (see, for instance, [1–3] and references therein). In the nuclear collisions at lower bombarding energies ($\sim$ 1 GeV/nucleon) particle production is strongly suppressed and fragmentation processes dominate. The experimentally observed fragment-size distributions also exhibit intermittent properties (see, for instance [4–6] and references therein). However, the intermittency of the nuclear fragmentation processes is substantially less known than intermittency of multiparticle production. Thermodynamic approach is actively used to study the intermittency phenomenon in multiparticle production [7–9] (and references therein) and it seems reasonable to use the multifractal thermodynamics to interpret also the experimental data on multifragmentation processes at nuclear collisions. In present note we introduce a critical points approach in the frames of multifractal thermodynamics for this purpose.

Another interesting question related to the nuclear multifragmentation is: whether there is some universality in the fragmentation processes. E.g., the intermittency in fragment-size distributions was recently also discovered (experimentally [10] and numerically [11]) in multifragmentation after macro-solids collisions. In this note we also present a comparison of the macro-solids fragmentation intermittency with the nuclear fragmentation intermittency to show universal character of the multifractal critical behavior of the multifragmentation processes.

Let us introduce some definitions. Suppose we have a random field X. Generalized scaling implies a scaling relationship between the moments of different order of X:

$$F_q \sim F_p^{\rho(q,p)} \tag{1}$$

where the moments F_p can be, for instance, ordinary ensemble moments $F_p \equiv \langle X^p \rangle$ ($\langle \ldots \rangle$ denotes a statistical average on ensemble) or scaled factorial moments [1] or conditional moments [4]. In present note we use all these types of moments to analyze existing data. It should be noted, that it was [4] where generalized scaling (1) for conditional moments was applied for the first time in context of the nuclear multifragmentation. For multiparticle production at high energies the generalized scaling using the scaled factorial moments was applied for the first time in [12].

From definition (1) one can see that:

$$F_q \sim F_p^{\rho(q,p)} \sim [F_r^{\rho(p,r)}]^{\rho(q,p)} \sim F_r^{\rho(p,r)\rho(q,p)}$$

(where representation $F_p \sim F_r^{\rho(p,r)}$ was used, cf. (1)). On the other hand, $F_q \sim F_r^{\rho(q,r)}$. Then, comparing these two representations of F_q, one obtains for the relative exponent $\rho(q,p)$ following equation

$$\rho(q,p)\rho(p,r) = \rho(q,r) \tag{2}$$

It follows from (2) that $\rho(q,p)$ can be represented by the form

$$\rho(q,p) = \frac{\chi(q)}{\chi(p)} \tag{3}$$

and that this representation is unique. A critical point p_c is defined by

$$\chi(p_c) = 0 \tag{4}$$

Let us recall the thermodynamic interpretation of multifractality [13,14]. Suppose that the total volume of a sample consists of a d-dimensional cube of size L. We divide this volume into N boxes of linear size l ($N \sim (L/l)^d$), each labeled by an index i, and construct for each box the measure of a field $\mu(\mathbf{x})$

$$\mu_i(l) = \int_{v_i} \mu(\mathbf{x})\, dv \tag{5}$$

where v_i is the volume of the ith box. The generalized dimension, D_p, is defined by

$$D_p = \lim_{(l/L)\to 0} \frac{\ln Z(p)}{(p-1)\ln(l/L)} \tag{6}$$

with the partition function $Z(p) = \sum_i \mu_i^p$. This implies the scaling behavior: $Z(p) \sim (l/L)^{\tau(p)}$ where $\tau(p) = D_p(p-1)$. Let us group the boxes with the singularity exponent α: $\mu_i(l) \sim (l/L)^{\alpha}$ (for small l/L) into a subset $S(\alpha)$. In the scaling hypothesis, the number of boxes dN_α needed to cover the subset $S(\alpha)$ should behave like: $dN_\alpha(l) = d\rho(\alpha)(l/L)^{-f(\alpha)}$. In these terms, $Z(p)$ can be represented as follows:

$$Z(p) \simeq \int d\rho(\alpha)(l/L)^{p\alpha - f(\alpha)} \tag{7}$$

In the limit $(l/L) \to 0$, the integral in (7) is dominated by the term $(l/L)^{\min_\alpha(p\alpha - f(\alpha))}$. From the definition of $\tau(p)$, one obtains $\tau(p) = \min_\alpha(p\alpha - f(\alpha))$. Thus $\tau(p)$ is obtained by Legendre transforming $f(\alpha)$. When $f(\alpha)$ and $\tau(p)$ are smooth functions, this relationship can be rewritten in the following way:

$$\tau(p) = p\alpha - f(\alpha), \qquad \frac{df}{d\alpha} = p \tag{8}$$

The thermodynamic interpretation of these relationships means that p can be interpreted as the inverse of temperature: $p \sim \beta = T^{-1}$ (taking the Boltzmann constant as 1) and the limit $(l/L) \to 0$ can be seen as the thermodynamic limit of infinite volume ($V = \ln(L/l) \to \infty$). Then by identifying $\alpha_i = \ln \mu_i / \ln(L/l)$ with the energy E_i (per unit of volume) of a microstate i, one can rewrite the partition function in the familiar form [15]: $Z(p) = \sum_i \exp(-\beta E_i)$. From the definition: $f(\alpha) \sim \ln N_\alpha((l/L))/\ln(L/l)$, the singularity spectrum $f(\alpha)$ is seen to play the role of the entropy (per unit of volume).

When one deals with thermodynamic phase transitions, it is important to choose relevant quantities for consideration of scale invariance in the vicinity of the critical point T_c (or β_c). In ordinary thermodynamics, this is $(T - T_c)$ [15]. However, for multifractal thermodynamics the relevant quantity is $(\beta - \beta_c)$ due to $p \sim \beta$ (see above and [16]). It is thus interesting to check a phase-transition-like behavior of the function $\chi(p)$ for p close to p_c (cf. [15, 16]), namely, $\chi(p) \sim (p - p_c)^\gamma$ where γ is some critical exponent.

It follows from definition of the normalized ordinary ensemble moments

$$F_p = \frac{\sum_{i=1}^{N} \mu_i^p}{N}$$

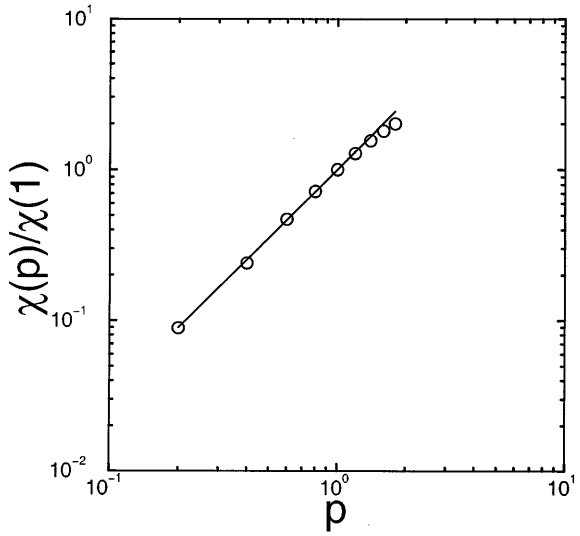

Fig. 1. $\chi(p)/\chi(1)$ against p in a log-log plot for the mass distribution of fragments in the experimental study of fragmentation of glass rods [10]. Straight line is the best fit which indicates agreement of the data with the critical representation (9)

that $\chi(0) = 0$ due to $F_0 = 1$ for this case. I.e. natural critical point for the ordinary ensemble moments is $p_c = 0$. For scaled factorial moments and for conditional moments $p_c = 1$ (see below). Let us start from a situation described by the ordinary ensemble moments (i.e. from situation with $p_c = 0$). In paper [10] a simple experiment on macro-solid fragmentation intermittency is described. Namely, brittle fracture of glass rods dropped onto the ground is studied experimentally with a focus on possible dependence of the distribution of fragment masses on the input energy that causes the fragmentation to occur. The input energy can be measured conveniently by the height from which the glass rod is dropped. Since the fragment-mass distribution $P(m)$ depends on the height of fall h the moments

$$F_p(h) = \frac{\sum_m P(m) m^p}{\sum_m P(m)} \tag{8}$$

also depend on h and $F_0 = 1$ from definition (8) (i.e. $p_c = 0$) in this case. The generalized scaling (1) was observed for these moments and the observed dependence $\chi(p)/\chi(1)$ is shown in Fig. 1. The log-log scales are used for comparison with the critical behavior

$$\chi(p)/\chi(1) = p^\gamma \tag{9}$$

which corresponds to $p_c = 0$. The straight line drawn in Fig. 1 indicates this behavior with $\gamma \simeq 1.49$ (about a possible meaning of the values of γ see discussion in the end of the note).

Let us now consider the critical multifractality at nuclear multifragmentation. In paper [5] the scaled factorial moments were used to analyze experimental data on the fragments mass distributions produced by interactions in nuclear emulsion, in which $^{197}_{79}Au_{118}$ nuclei of energy $E = 1$

GeV/nucleon breakup into fragments. Definition of the scaled factorial moments is

$$\langle F_q \rangle = M^{q-1} \langle \sum_{p=1}^{M} \frac{n_p(n_p-1)...(n_p-q+1)}{\langle N \rangle^q} \rangle \quad (10)$$

where $\langle N \rangle$ is the mean fragments multiplicity in the interval Δs (s is the charge of the fragments), with a particular partition of the region of interest Δs in M bins of size $\delta s = \Delta s/M$, n_p is the number of fragments in the pth bin and the brackets $\langle ... \rangle$ denote average over many events (authors of [5] also used some smoothing operation). It is shown in [5] that these (smoothed) factorial moments exhibit scaling

$$\langle F_p \rangle \sim (\Delta s/\delta s)^{\zeta_p} \quad (11)$$

where ζ_p is some function on p. Using the ordinary scaling (11) one can obtain for the generalized scaling exponents $\chi(p)$ relationship

$$\chi(p)/\chi(q) = \zeta_p/\zeta_q \quad (12)$$

From the definition $\zeta_1 = 0$ and, therefore, $\chi(1) = 0$ for the scaled factorial moments. It means that $p_c = 1$ for this case. Thus, the critical representation for $\chi(p)$ in a vicinity of the critical point $p_c = 1$ is

$$\chi(p)/\chi(2) = (p-1)^{\gamma} \quad (13)$$

(cf. critical representation (9) corresponding to ordinary ensemble moments). In Fig. 2 we show the data taken from [5] calculated using (12). The axes in this figure are chosen for comparison with the critical representation (13) (the straight line in this figure indicates $\gamma \simeq 1.43$).

Let us also use the so-called conditional moments which were introduced in [4] to study hot-nuclei multifragmentation. These (single-event) normalized moments are defined as follows

$$F_p^{(j)} = M_p^{(j)}/M_1^{(j)} \quad (14)$$

where

$$M_p^{(j)} = \sum_{s=1}^{s_{max}} s^p n^{(j)}(s) \quad (15)$$

and $n^{(j)}(s) = 0, 1, 2, ...,$ is the number of fragments of charge s appearing in the event j. From the definition $p_c = 1$ for these moments and critical representation (13) should be checked in a vicinity of this critical point. In a recent paper [6] the generalized scaling (1) for the conditional moments was calculated for experimental data on the diffractive excitation and electromagnetic dissociation of sulphur nuclei (^{32}S) at 200 GeV/nucleon (the fragmentation channel data were taken from [17,18]). In Fig. 3 the generalized scaling exponents are shown versus $(q-1)$ using log-log scales to check the critical relationship (13) (straight lines) for the diffractive excitation (circles) and for the electromagnetic dissociation (squares) data. The multifractal critical exponent $\gamma \simeq 0.86$ for the diffractive

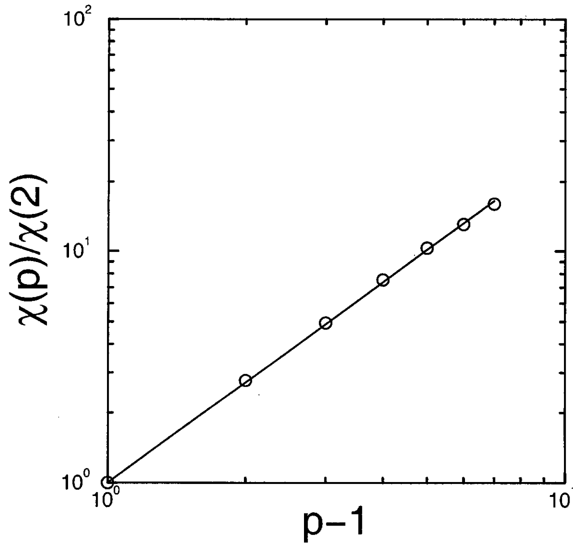

Fig. 2. $\chi(p)/\chi(2)$ against $(p-1)$ in a log-log plot for the fragment-size distribution at nuclear multifragmentation in nuclear emulsion, in which $^{197}_{79}Au_{118}$ nuclei of energy $E = 1$ GeV/nucleon breakup into fragments (data taken from [5]). Straight line is the best fit which indicates agreement of the data with the critical representation (13)

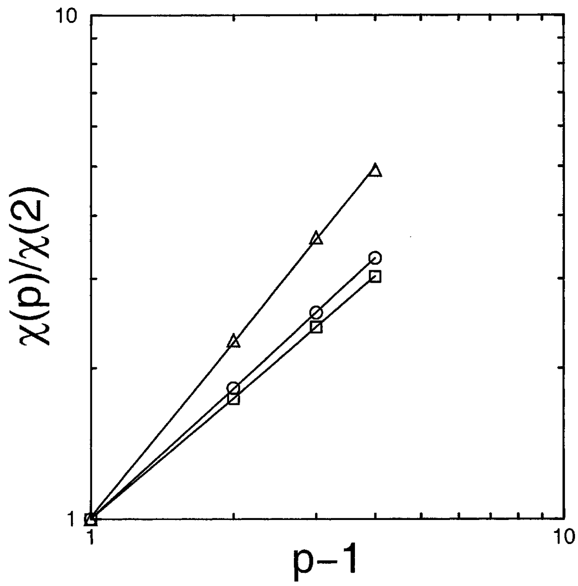

Fig. 3. $\chi(p)$ against $(p-1)$ in a log-log plot for the fragment-size distribution at nuclear multifragmentation for experimental data on the diffractive excitation (circles) and electromagnetic dissociation (squares) of sulphur nuclei (^{32}S) at 200 GeV/nucleon (the fragmentation channel data taken from [6], [17], [18]). The triangles correspond to mass distribution of fragments in the fragmentation by collision of two solid disks (the data taken from [11]). The straight lines are the best fit which indicate agreement with the critical representation (13)

excitation events and $\gamma \simeq 0.80$ for the electromagnetic dissociation events. The diffractive excitation process is a nuclear reaction while the electromagnetic dissociation process is an electromagnetic interaction. However, one can see that the multifractal critical fragmentation relationship is applicable to both these multifragmentation processes. Moreover, in a recent paper [11] fragmentation in collision of solids was simulated using a dynamical model of granular solids [11]. In this model, the solid consists of unbreakable and undeformable grains that are connected by elastic beams which can be broken according to a rule that takes into account of stretching and bending of the connections. The fragment mass distribution was found to depend on a dimensionless parameter η, which is the square root of the ratio of the collision and the binding energies. The authors of [11] used the normalized conditional moments (14), (15) to calculate the generalized scaling exponents. We show the data taken from [11] in Fig. 3 as well (triangles). One can see that these data also satisfies to the multifractal critical relationship (13) (with $\gamma \simeq 1.15$).

Finally, let us discuss several points which could be important for future investigations. The first point is the meaning of the various values for the critical exponent γ. In [19] a family of systems with infinite multifractal critical temperature was rigorously considered and a theory of multifractal phase transitions with *arbitrary* integer order was developed. For this special case $\chi(p) \sim p^\gamma$ in a vicinity of the critical point $p_c = 0$. It is suggested in [19] to relate the critical index γ to order of corresponding multifractal phase transition. In this approach, in particular, case $\gamma > 1$ corresponds to multifractal phase transition of a finite order (this order is calculated using the value of γ), while the case $\gamma < 1$ corresponds to multifractal phase transition of infinite order (in the terms of the theory developed in [19]). Now we don't know whether the model consideration performed in [19] can be applied to the general case investigated in present note. However, the idea to relate various values of the multifractal critical index to different orders of corresponding multifractal phase transitions could be useful for future investigations.

Furthermore, finite size effects may be quite important in the nuclear reactions. In particular, in finite systems universal critical behavior can be induced by finite size effects [20]. On the other hand, it was claimed in several theoretical works [21], [22] that for ordinary thermodynamics it is difficult to compare the exponents obtained for finite systems with the expected ones corresponding to the relevant universal class, due to the deformations induced by finite size effects. If there indeed exists relation between value of the multifractal critical exponent γ and order of corresponding multifractal phase transition [19], then one can expect that the problem with finite size effects (so difficult in ordinary thermodynamics) is not so crucial in the multifractal thermodynamics. This could also explain the fact that all (nuclear and macro-solid) data available in present time seem to be consistent with the critical multifractal representation.

The author is grateful E.S.C. Ching and F. Kun for providing the data, to E.S.C Ching, D. Stauffer and to N. Schörghofer for discussions and to Referee for comments and suggestions.

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